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TWO SUFFICIENT CONDITIONS FOR SUPERSOLVABILITY OF FINITE GROUPS

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ABSTRACT
In this paper two sufficient conditions for supersolvability of finite groups are given considering the maximal subgroups of Sylow subgroups of the group. An example is given to show that the conditions given are not necessary for supersolvability.

1. Introduction

It is well known that a finite group is nilpotent if all its Sylow subgroups are normal. In this note we consider finite groups for which we impose the condition that the maximal subgroups of Sylow subgroups are normal. We prove that such a group is supersolvable. Then we prove that supersolvability follows under the weaker assumption of π-quasinormality on the maximal subgroups of Sylow subgroups. An example is given to show that the group need not be supersolvable if we only require that the maximal subgroups of Sylow subgroups be subnormal instead of π-quasinormal. However, in this case the group will have the Sylow Tower Property and hence solvable.

2. Definitions and notations

1. A subgroup H of a group G is said to be *π-quasinormal* if HP is a subgroup of G for Sylow p-subgroups P of G for primes p in the set π.

2. A subgroup H of G is said to be *subnormal* in G if there exists a series of subgroups H_i of G such that

$$H = H_0 \lhd H_1 \lhd \cdots \lhd H_n = G.$$

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3. The group G is said to have Sylow Tower Property for the ordering of the primes dividing $|G|$ given by $p_1 > p_2 > \cdots > p_n$ provided

$$G_{p_1} \lhd G_{p_2} G_{p_1} \lhd \cdots \lhd G = G_{p_n} \cdots G_{p_1}$$

with G_{p_i} denoting the Sylow p_i-subgroup of G.

We use the following notations:
1. $\pi(G)$ is the set of prime divisors of $|G|$.
2. $M < \cdot G$ means M is a proper maximal subgroup of G.
3. $M \lhd \lhd G$ means M is a subnormal subgroup of G.

3. Main results

In this section we use the following two known results which we state as Lemmata.

LEMMA A (Itô [1]). *Let G be a finite group and H a nilpotent subnormal subgroup of G. Then G contains a nilpotent normal subgroup of G containing H.*

LEMMA B ([2]). *Let P be a Sylow p-subgroup of G. Suppose that P is cyclic and P is not contained in the derived subgroup of G. Then G has a normal p-complement.* (See lemma 12.10, page 105 of [2].)

THEOREM 1. *Let G be a finite group with the property that maximal subgroups of Sylow subgroups are normal in G. Then G is supersolvable.*

PROOF. Let N be the product of the maximal subgroups of Sylow subgroups of G. Then G/N is a group of square-free order and hence supersolvable. Clearly N is nilpotent since all its Sylow subgroups are normal in it. Hence G is solvable. Let M be any maximal subgroup of G. Therefore M has prime power index in G. Let $[G: M] = p_i^n$ where $|G| = \Pi_{i=1}^r p_i^{a_i}$. M is solvable and Sylow p_j-subgroups of M are Sylow p_j-subgroups of G for $j \neq i$. Solvability of M guarantees the existence of a subgroup of order $\Pi_{j=1, j\neq i}^r p_j^{a_j}$. Let this subgroup be H. Again, by solvability of M, we can find a Sylow p_i-subgroup M_{p_i} of M such that $M = HM_{p_i}$. Therefore $M_{p_i} \leqq P_0 < G_{p_i}$ and $P_0 \lhd G$, by hypothesis. Hence we have $M = HP_0$ and so $[G: M] = p_i$. So we have shown that all maximal subgroups of G have prime index in G and hence G is supersolvable by a well known theorem of Huppert.

THEOREM 2. *Let G be a finite group with the property that maximal subgroups of Sylow subgroups are π-quasinormal in G for $\pi = \pi(G)$. Then G is supersolvable.*

PROOF. Let P be any Sylow p-subgroup of G and let $P_0 < \cdot P$. By hypothesis P_0 is π-quasinormal in G and so $P_0 \lhd \lhd G$. P_0 is π-quasinormal in G implies that P_0G_q is a subgroup of G for every prime divisor q of $|G|$, where G_q denotes a Sylow q-subgroup of G. Since P_0 is subnormal in G, it is subnormal in P_0G_q and hence $P_0 \lhd P_0G_q$ for $q \neq p$. Therefore $G_q \leqq N_G(P_0)$ for every $q \neq p$. Since $P_0 < \cdot P$, we have that $P \leqq N_G(P_0)$. Therefore $P_0 \lhd G$. Now applying Theorem 1 we see that G is supersolvable.

We now replace the π-quasinormality in Theorem 2 with subnormality. Then the group need not be supersolvable. Consider the alternating group on 4 letters, denoted by A_4. Any subgroup of order 2 is a maximal subgroup of its Sylow 2-subgroup, subgroups of order 2 in A_4 are subnormal in A_4 and Sylow 3-subgroup of A_4 being of order 3 its maximal subgroups are trivial. Thus A_4 has the property that maximal subgroups of its Sylow subgroups are subnormal in A_4 and A_4 is not supersolvable. However, we have the following theorem.

THEOREM 3. *Let G be a finite group with the property that maximal subgroups of its Sylow subgroups are subnormal in G. Then G has the Sylow Tower Property for some ordering of the primes in $\pi(G)$ and hence G is solvable.*

PROOF. Applying Lemma A we see that either a Sylow subgroup is normal or its maximal subgroups are normal. If at least one of the Sylow subgroups of G is normal in G then factoring out that normal Sylow subgroup and applying induction to the factor group we see that the factor group has the Sylow tower property and hence G has the Sylow Tower Property. If none of the Sylow subgroups of G are normal in G, then by the earlier observation we have that all Sylow subgroups of G have their maximal subgroups normal in G and hence G is supersolvable by Theorem 1 and so G has the Sylow Tower Property.

The proof of Theorem 1 shows that under the assumptions there either a Sylow subgroup is normal or it is cyclic since when a maximal subgroup of a Sylow subgroup is normal it is the intersection of all the Sylow subgroups of G for that prime. Such a subgroup is called the core of that Sylow subgroup of G. Hence, when a Sylow subgroup is not normal it has a unique maximal subgroup, namely its core, and hence that Sylow subgroup is cyclic. Thus we see that when we assume that maximal subgroups of Sylow subgroups of a group are normal in the whole group then that group has the property that its Sylow subgroups are either normal or cyclic. In Theorem 1 we have shown that the group G is supersolvable if maximal subgroups of Sylow subgroups of G are normal in G. G is supersolvable implies that G' is nilpotent. So, if P is a non-normal Sylow subgroup of G then P is not contained in G'. Now applying Lemma B we see that G has a normal p-complement if P were a Sylow p-subgroup of G. Hence Theorem 1 leads to the interesting question as to what can be said about a group in which either Sylow p-subgroup or Sylow p-complement is normal for every prime divisor p of $|G|$. It is easy to verify that such a group has the Sylow Tower Property for some ordering of the primes in $\pi(G)$.

It can be remarked further that the condition $P_0 < \cdot P$ and $P_0 \lhd \lhd G$ can be weakened in Theorem 3 to the condition that $P_0 \lhd \lhd G$ and $P = \langle P_0, g \rangle$. In this case it can be proved that G is the extension of a nilpotent group by a metacyclic group and hence solvable. Indeed, $P_0 \lhd \lhd G$ and $P = \langle P_0, g \rangle$. Since $P_0 \lhd \lhd G$ using Lemma A of this paper there exists $K \lhd G$, K nilpotent and $P_0 \leqq K$. Hence the Sylow p-subgroup of K is normal in G and it contains P_0. So we can assume without loss of generality that K is a p-group. Therefore $P = \langle P_0, g \rangle = K\langle g \rangle$. Since this applies for all primes that divide the order of G we see that G has a nilpotent normal subgroup N, viz., the product of all the K's. Consider G/N. G/N has cyclic Sylow subgroups for all primes that divide the order of G/N since $P = K\langle g \rangle$. Now using lemma 12.9, page 104 of [2] we see that G is the extension of a nilpotent group by a metacycic group.

On the other hand if we require in Theorem 3 that second maximal subgroups of Sylow subgroups are subnormal in G instead of the maximal subgroups of Sylow subgroups, then we can show that G is solvable if A_5, the alternating group on 5 letters, is not involved in G and that G has the Sylow Tower Property for some ordering of the primes in $\pi(G)$ provided 6 does not divide the order of G. This latter condition is not sufficient as A_4 shows.

We now give an example to show that there are supersolvable groups without the property that maximal subgroups of its Sylow subgroups are normal thereby showing that the condition of Theorem 1 is only sufficient and not necessary.

Example. Let $G = \langle a, b;\, a^5 = 1, b^4 = 1, b^{-1}ab = a^2 \rangle$. G is supersolvable of order 20 is easy to see since the Sylow subgroups of G are cyclic. G has a subgroup of order 10, say H, generated by the elements a and b^2. H is normal in G since it has index 2 in G. This group G does not have a normal subgroup of order 2. So G does not have the property that the maximal subgroups of Sylow subgroups are normal in G.

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